Differentiation: Chain Rule

The **Chain Rule** is used when we want to differentiate a function that may be regarded as a composition of one or more simpler functions.
If our function $f(x) = (g \circ h)(x)$, where g and h are simpler functions, then the Chain Rule may be stated as

$$\boxed{f'(x) = (g \circ h)'(x) = (g' \circ h)(x)h'(x).}$$

There is also another notation which can be easier to work with when using the Chain Rule.
Let u be a function of x, i.e., $u(x)$ and let y be a function of u, i.e., $y(u)$ then the derivative of $y(u(x)) = f(x)$ is given by

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.}$$

Note that this second form of notation can still be used even if the question is framed in terms of $f(x)$ (unless you are specifically instructed otherwise) and this is what we will do below.

Example 1: Find the derivative of $f(x) = \sin(x^2)$.

Solution 1: In general the harder part of using the Chain Rule is to decide on what u and y are. The key is to look at the function we are given and see what it does to x 'first', or equivalently what the 'inner' function is, since this is what we will let our u be. In this case we see that the first thing f does to x is to square it, so we let $u = x^2$.

We now see that letting $y = f(x)$, we have $y = \sin(x^2)$ or $y = \sin u$ if we replace x^2 by u.

Next we find the derivatives $\frac{dy}{du}$ and $\frac{du}{dx}$:

$$\frac{dy}{du} = \cos u \qquad \text{and} \qquad \frac{du}{dx} = 2x.$$

Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\cos u)(2x) = \cos(x^2)(2x) = 2x\cos(x^2).$$

Thus

$$f'(x) = 2x\cos(x^2).$$

There are a couple of important points to note about the way we have presented our solution. Firstly there was no u in the question, it was something that we introduced in order to enable us to solve the problem, so it should not be in the solution, which should be in terms of x. Secondly the question asked for $f'(x)$, so we have given our solution in that form. If the question had asked for $\frac{dy}{dx}$ then we would have presented it as $\frac{dy}{dx} = 2x\cos(x^2)$.

Example 2: Find the derivative of $f(x) = \dfrac{5}{x^3 + x^2 + x + 1}$.

Solution 2: Note this is the same problem as Example 4 of the **Differentiation: Quotient Rule** worksheet.
In this case it might not be immediately apparent what the 'inner' function is. However if we write $f(x) = 5(x^3 + x^2 + x + 1)^{-1}$ using a rule of indices, then it becomes clear that we should let

$$u = x^3 + x^2 + x + 1 \qquad \text{and} \qquad y = 5u^{-1}.$$

Differentiating we obtain

$$\frac{dy}{du} = -5u^{-2} \qquad \text{and} \qquad \frac{du}{dx} = 3x^2 + 2x + 1.$$

Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left(-5u^{-2}\right)\left(3x^2 + 2x + 1\right) = -5(x^3 + x^2 + x + 1)^{-2}(3x^2 + 2x + 1).$$

Thus

$$f'(x) = -5(x^3 + x^2 + x + 1)^{-2}(3x^2 + 2x + 1) \qquad \text{or} \qquad f'(x) = -\frac{5(3x^2 + 2x + 1)}{(x^3 + x^2 + x + 1)^2}.$$

Example 3: Find the derivative of $f(x) = e^{\sin(3x)}$.

Solution 3: Similarly to Example 2, if we rewrite the function as $f(x) = \exp(\sin(3x))$ then it becomes more apparent that we should let

$$u = \sin(3x) \qquad \text{and} \qquad y = \exp(u) = e^u.$$

Differentiating we obtain

$$\frac{dy}{du} = e^u \qquad \text{and} \qquad \frac{du}{dx} = 3\cos(3x).$$

Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (e^u)(3\cos(3x)) = 3\cos(3x)e^{\sin(3x)}.$$

That is

$$f'(x) = 3\cos(3x)e^{\sin(3x)}.$$

Example 4: Find the derivative of $f(x) = \ln(\sin(x^2))$.

Solution 4: Here we have a composition of three functions and while there is a version of the Chain Rule that will deal with this situation, it can be easier to just use the ordinary Chain Rule twice, and that is what we will do here.

In fact we have already found the derivative of $g(x) = \sin(x^2)$ in Example 1, so we can reuse that result here. Of course if we did not have that result then we would first have to use the Chain Rule to find the derivative of the 'inner composition of functions'. So we let

$$u = \sin(x^2) \qquad \text{and} \qquad y = \ln u.$$

Differentiating we obtain

$$\frac{dy}{du} = \frac{1}{u} \qquad \text{and} \qquad \frac{du}{dx} = 2x\cos(x^2).$$

Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left(\frac{1}{u}\right)(2x\cos(x^2)) = \frac{2x\cos(x^2)}{u} = \frac{2x\cos(x^2)}{\sin(x^2)}.$$

That is

$$f'(x) = \frac{2x\cos(x^2)}{\sin(x^2)} \qquad \text{or} \qquad f'(x) = 2x\cot(x^2).$$